

Quantum Cube: A Toy Model of Qubit

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Representation of our knowledge depends on ways in which information is gained. We discuss a simple system whose 'ontic' state space gets reshaped due to a specific model of measurement and transformations. It is shown that the 'epistemic' description of the model faithfully reconstructs the characteristic quantum-like behavior of a certain set of states of a qubit.

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Description of a system crucially depends on how much, and in what way, information about its real state can be learned. Individuals with different perception capabilities may draw quite a different picture of the same underlying reality. This often occurs when limitations on gaining knowledge are imposed and one is to devise a consistent theory giving account of the phenomena observed in the constrained regime [1]. Since cognitive restrictions may affect the individual conception of reality a practical way to handle this situation is to carefully distinguish the concept of an *ontic state* (state of reality) and an *epistemic state* (state of knowledge). The latter, referring only to the knowledge actually available to the agent, captures an adequate image of reality relative to the agent's probing capabilities.

The distinction between the ontic and the epistemic standpoint acquires special significance in the context of the information theoretic approach to quantum foundations. Recent developments in the subject attach precise meaning to the ψ -*ontic* and ψ -*epistemic* models of the theory characterized by different interpretations of quantum states [2, 3]. Many results suggest that quantum states can be understood as states of knowledge. Strong evidence in favor of this view is given by concrete models providing analogues of various phenomena typically associated with strictly quantum mechanical characteristics. Most notable in this respect is the Spekken's toy model [2] reproducing a surprisingly large array of effects. However, despite qualitative resemblance it is not a constrained version of the theory. Sources of this dissimilarity have been well studied in the recent research [4–6], as well as some extensions have been proposed [7–9]. Moreover, various properties and structural constraints to be satisfied by ψ -*epistemic* reconstructions have been recognized, e.g. [10–14]. In this letter, we give an example of a simple classical system and discuss its epistemic description adjusted to a specific model of measurement and transformations. The model will be shown to be in full analogy to a certain set of states of a single qubit.

A good theory needs to account for measurement results performed on a system prepared and transformed according to some well specified collection of procedures. The latter, defining agent's perception capabilities, play an active role in shaping a theory. The '*ontic*' perspec-

tive, which assumes access to an exhaustive repertoire of investigation procedures, allows in principle to recover a faithful image of the (ontic) reality. However, constraints on how the system can be probed bounds one to adopt the '*epistemic*' view leaving its true nature in many cases suppressed as the description of the system is to account only for results of procedures that are at the agent's disposal. This distinction is crucial as, interestingly enough, various restrictions may lead to different theories whose shape, content and complexity is heavily contingent on ways in which information about the system is gained. In this letter, we discuss a simple classical model of a cube and show a transition from the ontic to the epistemic description induced by certain cognitive restrictions. We begin with the 'ontic' definition of the system and a brief discussion of a conventional probabilistic setup. Cognitive restrictions are introduced in two steps. First, we explicitly define model of a measurement which reveals partial information about the system at a cost of disturbance (it also serves as a simple preparation procedure). In the second step, we identify a limited set of transformations at agent's disposal, thereby extending means by which the system can be probed (and prepared). These restrictions, furnishing an airtight framework of possible agent's actions, will be further analyzed to build the 'epistemic' description of the system providing a toy analogue of a single qubit.

We will consider an elementary system with 8 possible (ontic) states $\Omega = \{\omega_1, \dots, \omega_8\}$. For future convenience let us represent the state space Ω as a cube, and depict the system in state ω_i as occupying the i -th vertex, see

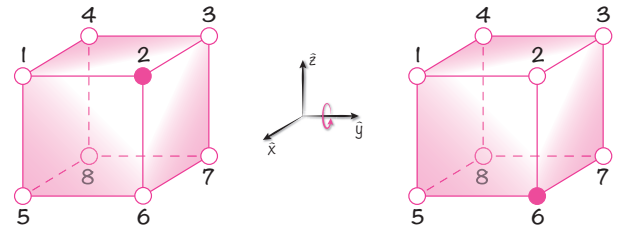


FIG. 1. Eight vertices of a cube represent ontic states of a system. The system in state ω_2 (on the left) can be transferred to state ω_6 (on the right) e.g. via rotation $R_{\hat{y}}(\frac{\pi}{2})$.

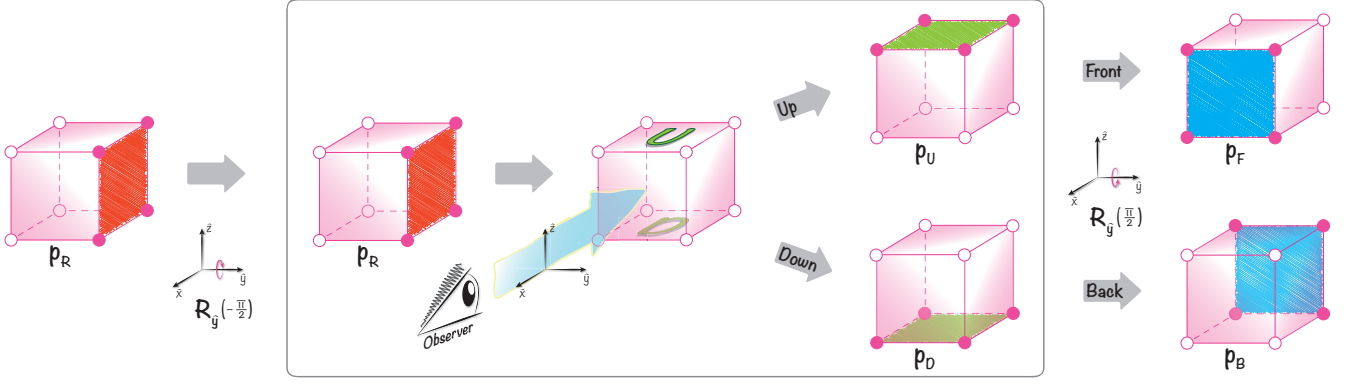


FIG. 2. Shaded areas of the cube depict equiprobable mixtures of the adjacent vertices representing ontic states of the system. Box in the middle illustrates the basic measurement M_1 which distinguishes only between the upper U and the lower L face of the cube and produces states \mathbf{p}_U and \mathbf{p}_D respectively. The whole picture demonstrates the effect of a general measurement procedure $M_T = T \circ M_1 \circ T^{-1}$ for $T = R_{\hat{y}}(\frac{\pi}{2})$ which discriminates between the front F and the back B face of the cube.

Fig.1. A standard probabilistic description consists in specifying the probability vector $\mathbf{p} = (p_1, \dots, p_8)^T$ which encapsulates all information about relative occurrences p_i of the ontic states ω_i in a statistical ensemble [15]. Vector \mathbf{p} belongs to a convex set $\Delta = \{ (p_1, \dots, p_8)^T : \sum_{i=1}^8 p_i = 1, p_i \geq 0 \}$ spanned by extremal states $\mathbf{p}_1, \dots, \mathbf{p}_8$ (where $(\mathbf{p}_i)_j = \delta_{ij}$) corresponding to the system definitely being in the ontic state $\omega_1, \dots, \omega_n$ respectively. Note that, given \mathbf{p} , probability p_i of finding the system in state ω_i is recovered by a simple formula $P_{\mathbf{p}}(i) = \mathbf{p}_i \cdot \mathbf{p}$, and transformations are implemented by stochastic maps $\mathbf{p} \rightarrow S\mathbf{p}$.

From the 'ontic' perspective any vector $\mathbf{p} \in \Delta$ is a valid probability state. For example, it can be prepared as a mixture of the ontic states ω_i . We note, however, that this tacitly assumes ontic states to be directly accessible, i.e. one is able to prepare systems in well defined states ω_i . Another possibility is to transform some other valid state to the desired one; this, in turn, claims sufficiently large repertoire of transformation procedures and initial states. Without going into further discussion we observe that the 'ontic' standpoint stipulates access to resources reach enough to prepare any state in $\mathbf{p} \in \Delta$.

However, in reality such a privileged situation may not obtain, with agent facing limitations on gaining information, preparing and processing the system. Below, we discuss a simple model with restricted measurement and transformation procedures which significantly affect the conventional picture, confining agent to essentially 'epistemic' – rather than 'ontic' – description of reality.

A primary role of measurement is to reveal information. Let a *basic measurement*, denoted by M_1 , be defined to answer the question whether a given elementary system is in one of the upper $U = \{\omega_1, \dots, \omega_4\}$ or lower $L = \{\omega_5, \dots, \omega_8\}$ states, and subsequently alter the system leaving it with equal probabilities in one of the four

compatible states, i.e.

$$\begin{aligned} \omega_1, \dots, \omega_4 &\xrightarrow{\text{UP}} \mathbf{p}_U = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, 0, 0, 0\right)^T \\ \omega_5, \dots, \omega_8 &\xrightarrow{\text{DOWN}} \mathbf{p}_D = \left(0, 0, 0, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)^T \end{aligned}$$

See Fig.2 (in the box) for a schematic illustration. This can be informally imagined as if an observer was looking at the cube representing the system from the \hat{x} direction distinguishing only between states on the upper and lower face, and at the same time jiggling the cube in the horizontal plane. In other words, the measurement discriminates states in the \hat{z} direction while randomizing the system in the $\hat{x}\text{-}\hat{y}$ plane.

Clearly, information gain in such a measurement comes at a cost of disturbance. Nonetheless, the above definition guarantees reproducibility of results on individual systems. As for an ensemble described by \mathbf{p} , measurement on a randomly chosen system will yield outcome $\mu = U, D$ with probability given by the formula $P_{\mathbf{p}}(\mu) = 4 \mathbf{p}_{\mu} \cdot \mathbf{p}$. Moreover, in the aftermath of the measurement (performed on each element of the ensemble) the state changes to a mixture $P_{\mathbf{p}}(U) \mathbf{p}_U + P_{\mathbf{p}}(D) \mathbf{p}_D$. Effectively it is a transformation which in a compact notation reads $\mathbf{p} \rightarrow M_1 \mathbf{p}$, where M_1 is a block diagonal matrix with two blocks of size four entirely filled with $\frac{1}{4}$'s, i.e. $(M_1)_{ij} = \frac{1}{4} [i, j \leq 4] + \frac{1}{4} [i, j > 4]$ in Iverson notation [16]. Note that \mathbf{p}_U and \mathbf{p}_D remain unaltered by the measurement.

Suppose that the system can be probed only with the measurement M_1 and no other transformations are available. Then the only states within agent's reach for preparing the system are \mathbf{p}_U and \mathbf{p}_D , thereby rendering $\Delta_0 = \{ \alpha \mathbf{p}_U + \beta \mathbf{p}_D : \alpha + \beta = 1, \alpha, \beta \geq 0 \}$ to be a maximal set of states at her disposal. That being so, the state space in this scenario is equivalent to a *classical bit* $\Delta_{\text{class}} = \{ (p_0, p_1) : p_0 + p_1 = 1, p_0, p_1 \geq 0 \}$, which readily follows from the replacement $\mathbf{p}_U \leftrightarrow (1, 0)$ and $\mathbf{p}_D \leftrightarrow (0, 1)$. We stress the fact that from the agent's perspective it is a complete description of 'reality' which

is perceived and tackled with her limited resources. This is to say that information coded in a vector $\mathbf{p} \in \Delta_0$ (or equivalently Δ_{class}) is just enough to account for all possible actions in hand, i.e. preparations (mixtures of measurement outputs), transformations (only the identity transformation $\mathbb{1}$) and measurements (only measurement M_1). Thus, we have argued that epistemic description of the system, as seen from the the agent's perspective, is effectively that of a classical bit.

This preliminary example shows a way of operational implementation of cognitive restrictions, and briefly illustrates how to effectively proceed from the ontic to the epistemic standpoint. We develop this line of thought in the next section to see how it accommodates transformations.

Let us extend the above setup to include rotations of the cube in Fig.1 through angle $\frac{\pi}{2}$ about axes \hat{x} , \hat{y} , \hat{z} and combinations thereof. Here, by rotation of the system we mean the associated permutation of its ontic state space Ω , e.g. $R_{\hat{x}}(\frac{\pi}{2})$, $R_{\hat{y}}(\frac{\pi}{2})$, $R_{\hat{z}}(\frac{\pi}{2})$ represent permutations (1562)(3487), (1584)(2673), (1234)(5678) respectively. The full set of such *transformations*, denoted further by $\mathcal{R}_{\pi/2}$, consists of 24 elements which form a group of rotational symmetries of a cube [17]. Of course, this is still a limited set of transformations if compared with all conceivable mappings, yet it is more than in the foregoing discussion of the basic measurement M_1 with the trivial set of transformations $\mathcal{R}_0 = \{\mathbb{1}\}$.

Note that a constrained set of transformations presents another kind of cognitive restrictions affecting agent's inquiries about the system. We remark that this set usually carries additional structure which brings into play insightful group and symmetry concepts. This explains our choice of geometrical representation of the state space of the system Ω by a cube whose symmetries aptly capture properties of the transformation set $\mathcal{R}_{\pi/2}$.

Extension of the allowable set of transformations (from \mathcal{R}_0 to $\mathcal{R}_{\pi/2}$) improves agent's toolkit for probing the system. This comes from the fact that measurements and preparations may combine with transformations. Below we discuss the epistemic description of the system as seen by the agent, now, quipped with a richer set of tools.

Firstly, the system prepared as the outcome of measurement M_1 in state \mathbf{p}_U or \mathbf{p}_D can be transformed to one of the four new states:

$$\begin{aligned} \mathbf{p}_L &= (\frac{1}{4}, 0, 0, \frac{1}{4}, \frac{1}{4}, 0, 0, \frac{1}{4})^T, & \mathbf{p}_R &= (0, \frac{1}{4}, \frac{1}{4}, 0, 0, \frac{1}{4}, \frac{1}{4}, 0)^T, \\ \mathbf{p}_F &= (\frac{1}{4}, \frac{1}{4}, 0, 0, \frac{1}{4}, \frac{1}{4}, 0, 0)^T, & \mathbf{p}_B &= (0, 0, \frac{1}{4}, \frac{1}{4}, 0, 0, \frac{1}{4}, \frac{1}{4})^T. \end{aligned}$$

These states correspond to the system being with equal probabilities in one of the ontic states in the respective sets $L = \{\omega_1, \omega_4, \omega_5, \omega_8\}$, $R = \{\omega_2, \omega_3, \omega_6, \omega_7\}$, $F = \{\omega_1, \omega_2, \omega_5, \omega_6\}$, $B = \{\omega_3, \omega_4, \omega_7, \omega_8\}$. The sets U , D , L , R , F and B coincide with the faces of the cube (up, down, left, right, front and back), as depicted in Fig.3 on the left. Note that each state can be obtained in more

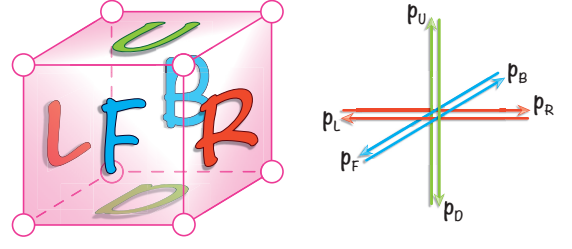


FIG. 3. On the left, faces of the cube corresponding to states \mathbf{p}_U , \mathbf{p}_D , \mathbf{p}_L , \mathbf{p}_R , \mathbf{p}_F and \mathbf{p}_B . On the right, vectors representing pairs of opposite faces tested in a general measurement procedure.

than one way, e.g. \mathbf{p}_F obtains from \mathbf{p}_U by rotation $R_{\hat{y}}(\frac{\pi}{2})$ or by rotations $R_{\hat{x}}(-\frac{\pi}{2}) \circ R_{\hat{y}}(\frac{\pi}{2})$ (equivalent to the rotation through $3\pi/2$ about the diagonal joining vertices 1 and 7). Since transformations from the set $\mathcal{R}_{\pi/2}$ just shuffle faces of the cube, then \mathbf{p}_U , \mathbf{p}_D , \mathbf{p}_L , \mathbf{p}_R , \mathbf{p}_F and \mathbf{p}_B is an exhaustive collection of states obtainable in this way. Hence, the most general states that can be prepared by the agent belong to the convex set

$$\Delta_{\pi/2} = \left\{ \sum_{\mu} p_{\mu} \mathbf{p}_{\mu} : \sum_{\mu} p_{\mu} = 1, p_{\mu} \geq 0 \right\}, \quad (1)$$

where $\mu = U, D, L, R, F, B$.

Secondly, each transformation $T \in \mathcal{R}_{\pi/2}$ has a canonical representation as an 8x8 permutation matrix $T_{ij} = \delta_{\sigma(i)j}$, where σ is the permutation of the vertices of the cube induced by rotation T . It readily carries over into the probabilistic description of the system whose state transforms as

$$\mathbf{p} \rightarrow T \mathbf{p}. \quad (2)$$

Since $\Delta_{\pi/2}$ is mapped into itself it remains the maximal set of states accessible to the agent.

Thirdly, a richer set of transformations provides subtler means for probing the system. A measurement preceded by a transformation reveals different information than a bare measurement M_1 . Therefore, for each $T \in \mathcal{R}_{\pi/2}$ we define a new measurement procedure

$$M_T = T \circ M_1 \circ T^{-1}, \quad (3)$$

which consists of the preparatory phase T^{-1} , the measurement M_1 furnishing the outcome, and the closing transformation preparing the output; see Fig.2. Although there are 24 such procedures, some of them are equivalent and we get only 6 essentially different measurements. This readily follows from the observation that the measurement M_1 distinguishes between the upper U and the lower D face of the cube which, due to preparatory rotation T^{-1} , are replaced by the left L and the right R , or the front F and the back B face (depending on T). Hence, the measurement M_T answers the question on which of the chosen pair of the opposite faces the system resides; e.g. $T = \mathbb{1}, R_{\hat{x}}(-\frac{\pi}{2}), R_{\hat{y}}(-\frac{\pi}{2}), R_{\hat{y}}(-\pi)$ test for (U, D) , (L, R) , (F, B) , (D, U) respectively. Since $\mathcal{R}_{\pi/2}$

are rotational symmetries of the cube there are only 6 different procedures which distinguish between elements of the pairs (U, D) , (D, U) , (L, R) , (R, L) , (F, B) , (B, F) . It is convenient to associate with these pairs vectors \mathbf{p}_U , \mathbf{p}_D , \mathbf{p}_L , \mathbf{p}_R , \mathbf{p}_F , \mathbf{p}_B respectively; see Fig.3 on the right. Then, probability of outcome ν in the measurement \mathbf{p}_μ performed on a system described by \mathbf{p} is given by the formula

$$P_{\mathbf{p}}(\nu) = 4 \mathbf{p}_\nu \cdot \mathbf{p}, \quad (4)$$

where ν takes only two values in accord with the chosen measurement \mathbf{p}_μ , e.g. for $\mu = L$ we have $\nu = L, R$. Note that the system gets altered in the aftermath of the measurement which, due to the closing part in the definition Eq.(3), leaves it with equal probabilities in one of the four compatible states, i.e. the output state is \mathbf{p}_ν if the outcome was ν ; see Fig.2. Observe that such defined measurements performed on individual systems are reproducible. Finally, we remark that we have discussed measurement M_T defined in Eq.(3) from the active point of view. An equivalent approach, from the passive point of view, would consist in leaving the system untouched and rotating the observer by $T \in \mathcal{R}_{\pi/2}$, thereby changing the axis along which the states are discriminated.

The above discussion presents a complete epistemic description of the system as seen by the agent equipped with a limited probing toolkit which consists only of the measurement M_1 and the collection of transformations $\mathcal{R}_{\pi/2}$. We have shown that the agent is entirely confined to the subspace $\Delta_{\pi/2}$ in a sense that it is just enough information to account for all her actions (i.e. possible preparations, transformations and measurements). Interestingly, this framework has a natural embedding in a familiar Hilbert space setup of a qubit. To see this consider the following correspondence between the probability vectors \mathbf{p}_μ and the pure states of a single qubit:

$$\begin{aligned} \mathbf{p}_U &\leftrightarrow |0\rangle\langle 0| & \mathbf{p}_F &\leftrightarrow |+\rangle\langle +| & \mathbf{p}_L &\leftrightarrow |+\rangle\langle +| \\ \mathbf{p}_D &\leftrightarrow |1\rangle\langle 1| & \mathbf{p}_B &\leftrightarrow |-\rangle\langle -| & \mathbf{p}_R &\leftrightarrow |-\rangle\langle -| \end{aligned}$$

where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ and $|\pm i\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle)$. This furnishes a one-to-one mapping from $\Delta_{\pi/2}$ into the following subset of density operators

$$\Phi = \left\{ \sum_{\kappa} p_{\kappa} |\kappa\rangle\langle \kappa| : \sum_{\kappa} p_{\kappa} = 1, p_{\kappa} \geq 0 \right\}, \quad (5)$$

where $\kappa = 0, 1, \pm, \pm i$. Explicitly, it is defined by

$$\Delta_{\pi/2} \ni \mathbf{p} = \sum_{\mu} p_{\mu} \mathbf{p}_{\mu} \longleftrightarrow \sum_{\kappa} p_{\kappa} |\kappa\rangle\langle \kappa| = \rho \in \Phi,$$

with the obvious replacement $\mu \leftrightarrow \kappa$ given above. We note that, in spite of non unique decomposition of states in the bases \mathbf{p}_{μ} or $|\kappa\rangle\langle \kappa|$, one can check by direct calculation that this mapping is well defined.

Hence, we may faithfully replace all the states $\Delta_{\pi/2}$ by their counterparts in Φ . The shape of Φ is that of the octahedron inscribed in the Bloch ball representing states

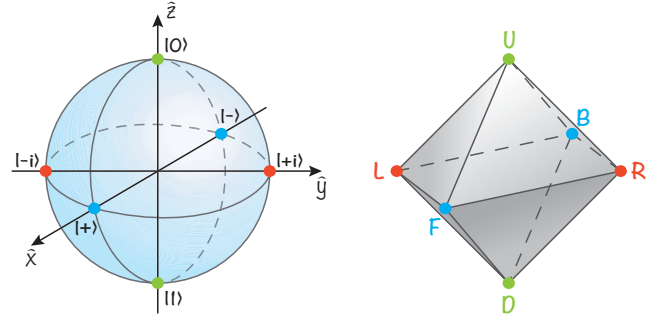


FIG. 4. On the left, the Bloch ball representing state of a single qubit. On the right, the octahedron Φ providing the equivalent Hilbert space description of the toy model (corresponds bijectively to the set of epistemic states $\Delta_{\pi/2}$).

of a single qubit, see Fig.4. It is a convex set spanned by the pure states $|\kappa\rangle\langle \kappa|$. This analogy smoothly extends to recover other aspects of the quantum formalism in our toy model. Accordingly, we get a correct account of measurements now described by the projectors $|\kappa\rangle\langle \kappa|$. In this representation probabilities of outcomes are calculated via the familiar formula $P_{\rho}(\kappa) = \langle \kappa | \rho | \kappa \rangle$ which replaces Eq.(4), and after the measurement state of the system updates according to the usual (quantum) projective rule. It is readily checked for the extremal states for which probabilities take only values 0, 1 or $\frac{1}{2}$, and by virtue of linearity directly extends to the whole set Φ . Let us further observe that a transformation $T \in \mathcal{R}_{\pi/2}$ in analogous manner rotates the cube in Fig.3 as it rotates the octahedron in Fig.4 (notice that these are dual solids [17], i.e. have the same set of symmetries). From the geometric picture we infer that transformations in $\mathcal{R}_{\pi/2}$ are represented by unitary maps transforming the octahedron Φ into itself, i.e. $\rho \rightarrow U\rho U^\dagger$ which takes the place of Eq.(2). In explicit form one recovers a standard (projective) representation of $\mathcal{R}_{\pi/2}$ in \mathbb{C}^2 , i.e. $R_{\hat{x}}(\frac{\pi}{2}) \leftrightarrow \frac{1}{\sqrt{2}}(\mathbb{1} - i\hat{\sigma}_x)$, $R_{\hat{y}}(\frac{\pi}{2}) \leftrightarrow \frac{1}{\sqrt{2}}(\mathbb{1} - i\hat{\sigma}_y)$, $R_{\hat{z}}(\frac{\pi}{2}) \leftrightarrow \frac{1}{\sqrt{2}}(\mathbb{1} - i\hat{\sigma}_z)$, etc. Thus the subset of states Φ together with the usual quantum rules provide equivalent description of the model discussed above.

In conclusion, we emphasize that the epistemic description of the toy model fully reconstructs a constrained version of a single qubit. Observe that the quantum-like behavior is a direct consequence of the interplay of two kinds of cognitive restrictions. While the measurement introduces a fundamental trade-off between information gain and state disturbance, it is the structure of the transformation set which brings in appropriate group and symmetry aspects. Conceptually, the underlying premise is that the only way to find out about the world is to interact with it (i.e. via measuring or manipulating it). This scheme furnishes a new approach to building and analyzing epistemic models. In particular, it should be possible to construct analogues of multipartite quantum systems.

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